Resonant transmission of a soliton across an interface between two Toda lattices

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The transmission of a single soliton is investigated numerically across an interface between two Toda lattices which are connected by a harmonic spring. We find that a resonant transmission of the soliton occurs when the spring constant of the harmonic spring is adjusted properly. Furthermore, when the amplitude of the incident soliton is large, the soliton transmission coefficient exhibits a local minimum which is due to an emergence of localized waves around the harmonic spring. We propose an experimental test of the results by using a nonlinear *LC* circuit.

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In the past decades various aspects of the soliton have been investigated and the application of the soliton has been developed on many fields. One of the most important applications is the optical soliton communication in optical fibers [1]. In the realistic optical soliton communication system, the soliton propagates through several fiber segments joined by a fiber splice and the soliton loses its own energy at the interfaces between the fiber segments [2]. When a soliton passes an interface between two segments which support the soliton, the soliton is scattered by the interface and the transmittance of the soliton would shrink.

In this paper, we investigate numerically the transmission of a soliton across an interface between nonlinear media. We treat the Toda lattice as a representative nonlinear medium which supports the propagation of soliton. It should be noted that the propagation of the soliton in the Toda lattice can be realized by a nonlinear *LC* circuit [3,4] and our numerical results can be tested experimentally as discussed later. We use a harmonic spring for a connector of the two Toda lattices. For simplicity, we consider two identical Toda lattices. The harmonic spring is not suited for the propagation of the soliton and can be interpreted as an impurity. It is known that localized waves can exist at a light mass impurity or a strong coupling impurity in a harmonic lattice and the Toda lattice [5,6]. We show that the localized waves also exist at the harmonic spring in our model. We try to enhance the transmittance of the incident soliton by controlling the property of the harmonic spring. It is found that there is a resonant transmission of the soliton for each amplitude of the incident soliton.

We consider two Toda lattices [7] connected by a harmonic spring. The Hamiltonian of our model is given by

$$H = \sum_n \left[\frac{p_n^2}{2m} + \phi_n(u_n) \right], \tag{1}$$

$$u_n = q_{n+1} - q_n, \tag{2}$$

where p_n and q_n are the momentum and displacement of a particle on site n, respectively, and m is the mass of the particle. ϕ_n is the interaction potential between the particles on sites n and $n+1$ given by

$$\phi_n(u_n) = \begin{cases} \dfrac{a}{b}[\exp(-bu_n) + bu_n - 1] & \text{for } n \leqslant -1 \text{ and } n \geqslant 1, \\ \dfrac{1}{2}Ku_n^2 & \text{for } n = 0, \end{cases} \tag{3}$$

where K is a spring constant of the harmonic spring, b is a controlling parameter of the nonlinearity and ab represents a spring constant of the Toda springs.

We introduce dimensionless variables defined by

$$\tau = \left(\frac{ab}{m}\right)^{1/2} t, \quad P_n = \left(\frac{b}{ma}\right)^{1/2} p_n, \quad Q_n = bq_n, \tag{4}$$

where τ is the dimensionless time. The equation of motion is reduced to

$$\frac{d^2 Q_n}{d\tau^2} = -\frac{d}{dQ_n}[\Phi_n(U_n) + \Phi_{n-1}(U_{n-1})], \tag{5}$$

$$\Phi_n(U_n) = \begin{cases} \exp(-U_n) + U_n - 1 & \text{for } n \leqslant -1 \text{ and } n \geqslant 1, \\ \dfrac{1}{2}\kappa U_n^2 & \text{for } n = 0, \end{cases} \tag{6}$$

where $\kappa = K/(ab)$ and $U_n = Q_{n+1} - Q_n$.

At time $\tau = 0$, we prepare a single-soliton

$$U_n = \ln\{1 + \omega_0^2 \operatorname{sech}^2[k_0(n - n_0) - \omega_0 \tau]\}|_{\tau=0}, \tag{7}$$

as an incident wave (Fig. 1), where k_0 is regarded as a wave number of the soliton, $\omega_0 = \sinh k_0$ and n_0 denotes the location of the soliton at $\tau = 0$. We set $n_0 \ll 0$ so that the incident soliton is at far left from the harmonic spring. The energy of

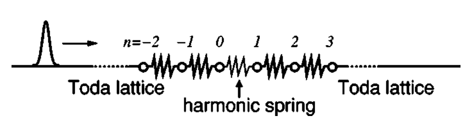

FIG. 1. The perspective of an incident soliton on two Toda lattices connected by a harmonic spring.

the incident soliton normalized by a/b is given by

$$E_0 = 2(\sinh k_0 \cosh k_0 - k_0). \tag{8}$$

When the incident soliton passes the harmonic spring, the incident wave is divided into three waves: transmitted, reflected and localized waves. The transmitted waves consist of a large soliton at the front and many small waves following it; we call them the frontier soliton and ripples, respectively. We use the frontier soliton transmission coefficient [8,9] given by

$$T_1 = E_1/E_0, \tag{9}$$

as a measure of transmission, where E_1 is the energy of the frontier soliton. In order to investigate the localized waves around the harmonic spring, we define the temporal localized energy by

$$L(\tau) = \sum_{n=-2}^{3} h_n(\tau), \tag{10}$$

where the maximum and minimum lattice numbers are selected so that $L(\tau)$ does not fluctuate for large τ and $h_n(\tau)$ is the normalized energy density given by

$$h_n(\tau) = [P_n^2 + \Phi_n(U_n) + \Phi_{n-1}(U_{n-1})]/(2E_0). \tag{11}$$

As is shown in the following numerical results, $L(\tau)$ approaches an asymptotic value for large τ and we define a localized energy L by the asymptotic value of $L(\tau)$. We also define a reflected energy R and an energy of the ripples T_{ri} by asymptotic values of

$$R(\tau) = \sum_{n=\text{left end}}^{-3} h_n(\tau), \tag{12}$$

$$T_{\mathrm{ri}}(\tau) = \left[\sum_{n=4}^{\text{right end}} h_n(\tau)\right] - T_1, \tag{13}$$

respectively. Since the total energy is a conserved quantity, the summation of the normalized energy density $h_n(\tau)$ is conserved and the following sum rule,

$$T_1 + L + R + T_{\mathrm{ri}} = 1, \tag{14}$$

is satisfied. We investigate these quantities for various sets of parameters in the following discussion.

We integrate Eq. (5) numerically using a third order bilateral symplectic algorithm [10]. We set integration step $\Delta\tau \leqslant 0.01$ so that total energy does not deviate from the value of the initial state. We impose fixed boundary conditions at both ends of entire sites. Since we prepare a sufficiently long lattice and finish the numerical calculation before waves reach the end of the lattice, the boundary condition does not affect the results.

In Fig. 2, we show the normalized energy density $h_n(\tau)$ as a function of n and τ for the spring constant of the harmonic spring $\kappa=(a)$ 38.54 and (b) 7.776, keeping the wave number of the incident soliton $k_0=2.5$ fixed. Figure 2(a) shows that the incident soliton is divided into frontier soliton, ripples, localized wave and reflected wave. In Fig. 2(b), since we chose the spring constant κ properly, the incident soliton completely passes the harmonic spring.

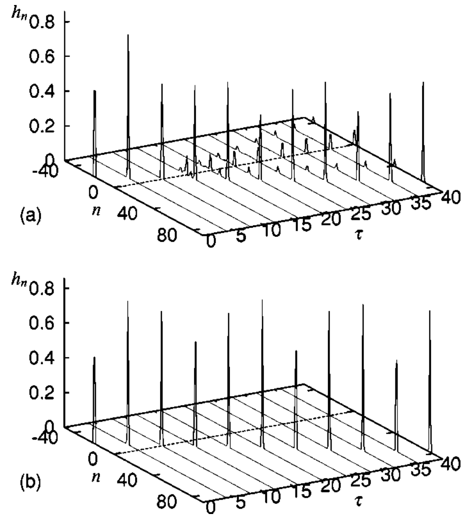

FIG. 2. The spatiotemporal evolution of energy density $h_n(\tau)$ which is normalized by total energy. τ is dimensionless time. The wave number k_0 of the incident soliton is $k_0=2.5$. The spring constants κ of the harmonic spring are (a) $\kappa=38.54$ and (b) $\kappa=7.776$.

We obtained T_1 for various spring constants κ of the harmonic spring, keeping the wave number k_0 of the incident soliton fixed. Figure 3 shows the κ dependence of T_1 for $k_0=1$, 1.5, 2 and 2.5. It can be seen from this figure that T_1 has a maximum as a function of κ. The maximal value is $T_1 \simeq 1$; in other words, there are no reflected waves and no localized waves and the incident soliton completely passes through the interface. We can regard this as a resonance between the soliton and the harmonic spring, which occurs at the spring constant of the harmonic spring chosen properly for each k_0. It can also be seen from Fig. 3 that when $k_0 \gtrsim 1.5$, T_1 has a local minimum as a function of κ. We show later that the local minimum is due to emergence of the localized waves around the harmonic spring. We note that in the limit of small κ, the particle at site 0 is disconnected from the other side of the system. In this case the incident soliton is reflected completely and T_1 vanishes at $\kappa=0$ since the

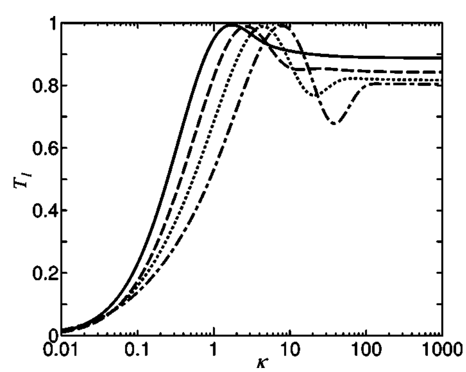

FIG. 3. The frontier soliton transmission coefficient T_1 as a function of the spring constant κ of the harmonic spring in the logarithmic scale. The wave numbers of the incident soliton are $k_0=1$(—), 1.5(---), 2(···) and 2.5(- · -).

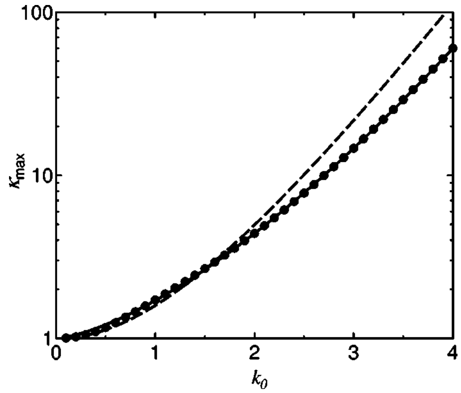

FIG. 4. The spring constants $\kappa_{\max}$ of the harmonic spring at the maximum of T_1 (●) are plotted as a function of the wave number k_0 of the incident soliton. The solid curve is the fit by Eq. (15). The broken curve is the approximant given by Eq. (18).

particle at site 0 behaves as a free boundary condition. On the other hand, in the limit of large κ, the particles at sites 0 and 1 are strongly connected with each other. Since these two particles behave as one heavy particle, T_1 approaches a finite asymptotic value as can be seen in Fig. 3.

We denote the κ at the maximum of T_1 by $\kappa_{\max}$. The $\kappa_{\max}$ depends on k_0 and is plotted against k_0 in Fig. 4. It can be seen from this figure that $\kappa_{\max}$ increases with the increase of k_0. The numerical result can be well fitted by a stretched exponential function:

$$\kappa_{\max}(k_0) \simeq \exp(\alpha k_0^{\beta}), \tag{15}$$

with $\alpha=0.5360$ and $\beta=1.467$. The result of the fit is also shown in Fig. 4.

For a qualitative understanding of the resonant transmission, we estimate the $\kappa_{\max}$ as follows. Equation (5) can be rewritten as

$$\frac{d^2 Q_n}{d\tau^2} = -\exp(-U_n) + \exp(-U_{n-1}) + R_n, \tag{16}$$

$$R_n = (\delta_{n,0} - \delta_{n,1})[\exp(-U_0) - 1 + \kappa U_0], \tag{17}$$

where R_n denotes residual. The frontier soliton transmission coefficient T_1 is expected to be close to 1 if the residual R_n is small. We approximate U_0 in Eq. (17) by the incident soliton and we choose κ so that $R_n=0$, where $|U_0|$ is maximum. We obtain an approximate expression for $\kappa_{\max}$ given by

$$\kappa_{\max}(k_0) = \frac{\sinh^2 k_0}{\ln(1+\sinh^2 k_0)}. \tag{18}$$

In Fig. 4, we compare this approximation for $\kappa_{\max}$ with the numerical result. It can be seen from this figure that the numerical result is well fitted by the approximate expression (18) when $k_0 \lesssim 1.7$. The discrepancy for large k_0 is due to the oversimplification of the approximation.

In order to investigate the localized waves around the harmonic spring, we obtain the temporal localized energy $L(\tau)$ defined by Eq. (10) and plot it in Fig. 5 as a function of time τ for $\kappa=3$ and 30 keeping $k_0=2.5$ fixed. The abrupt change

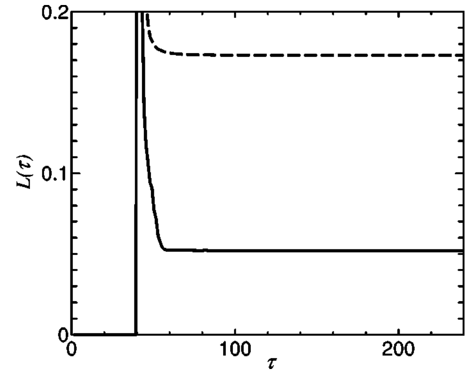

FIG. 5. The temporal localized energy $L(\tau)$ is plotted as a function of dimensionless time τ for $\kappa=3$(—) and 30(---). The wave number of the incident soliton is $k_0=2.5$.

between $\tau \simeq 40$ and 50 indicates that the soliton passes the harmonic spring. As can be seen from this figure, $L(\tau)$ does not change significantly when $\tau \gtrsim 80$ and we conclude that stable localized waves exist for certain values of parameters in our model. We note that $L(\tau)$ for other κ and k_0 behaves similarly. Therefore we can use the asymptotic value of $L(\tau)$ as the localized energy L.

We show the localized energy L, the frontier soliton transmission coefficient T_1, the reflected energy R and the energy of the ripples T_{ri} as functions of κ for $k_0=2.5$ in Fig. 6. Since there is the sum rule (14), the above four quantities are not independent. The frontier soliton transmission coefficient $T_1 \simeq 1$ at the $\kappa_{\max}$, and, as a result, the quantities R, L and T_{ri} are almost zero as mentioned above. It can be seen from this figure that L exhibits a local maximum at $\kappa \simeq 35$ and T_1 is a local minimum there. Since the quantities R and T_{ri} do not dramatically change around $\kappa=35$, the local minimum of T_1 is due to the existence of the peak of the localized energy L. On the other hand, although there is a hump of L between $\kappa=1$ and 8, T_1 decreases smoothly with the decrease of κ in this region of κ. Since T_{ri} takes large value there and R increases rapidly with the decrease of κ around $\kappa=1$ and the change of L is counterbalanced by the change of T_{ri} and R, the hump of L does not have much influence on the shape of the T_1 near $\kappa=1$. We note that the localized waves disappear

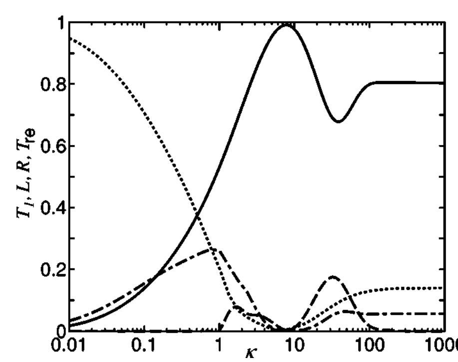

FIG. 6. The frontier soliton transmission coefficient T_1(—), the localized energy L(---), the reflected energy $R(\cdots)$ and the energy of the ripples T_{ri}(- · -) are plotted against κ. The wave number of the incident soliton is $k_0=2.5$.

in $\kappa < 1$ and $\kappa \gg 1$. The former corresponds to a weak coupling impurity and the latter corresponds to a heavy mass impurity as mentioned above.

It is well known that the Toda lattice can be realized using a *LC* ladder type electrical circuit [4]. The capacitance of the capacitor in the *LC* circuit depends on the applied voltage and is given by

$$C(V) = \frac{Q(V_0)}{F(V_0) + V - V_0}, \tag{19}$$

where $F(V_0)$ and $1/Q(V_0)$ are constants which correspond to the parameters of the Toda springs a and b in Eq. (3), respectively, and V_0 is the bias voltage. We found that our model can be constructed by replacing one of the nonlinear capacitors in the *LC* circuit with a capacitor whose capacitance is given by

$$C' = 1/\alpha, \tag{20}$$

where α is a constant corresponding to the spring constant of the harmonic spring K in Eq. (3).

We have investigated numerically the transmission of a soliton between two Toda lattices connected by a harmonic spring. It is shown that the transmittance of the soliton can be enhanced by controlling the property of the connector. We have suggested that our numerical results can be tested by using the nonlinear *LC* circuit.

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